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Inverse eigenvalue problems for linear complementarity systems

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ABSTRACT

Traditionally an inverse eigenvalue problem is about reconstructing a matrix from a given spectral data. In this work we study the set of real matrices A of order n such that the linear complementarity system

$$x \geqslant \mathbf{0}, \quad Ax - \lambda x \geqslant \mathbf{0}, \quad \langle x, Ax - \lambda x \rangle = 0$$

holds for prescribed pairs $(x_1, \lambda_1), \ldots, (x_p, \lambda_p)$. The analysis of this new type of inverse eigenvalue problem differs substantially from the classical one.

1. Introduction

The space $\mathbb{R}^n$ is equipped with the standard inner product $\langle \cdot, \cdot \rangle$ and its associated norm. The bold faced symbol $\mathbf{0}$ indicates a zero vector of appropriate dimension and $x \geqslant \mathbf{0}$ means that each entry of x is nonnegative. Let $\mathbb{M}_n$ denote the space of real matrices of order n.

Recall that a *Pareto eigenvalue* of $A \in \mathbb{M}_n$ is a scalar $\lambda \in \mathbb{R}$ such that the linear complementarity system

$$x \geqslant \mathbf{0}, \quad Ax - \lambda x \geqslant \mathbf{0}, \quad \langle x, Ax - \lambda x \rangle = 0 \tag{1}$$

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admits a nonzero solution $x \in \mathbb{R}^n$. By a positive homogeneity argument, there is no loss of generality in imposing the normalization condition $\langle \mathbf{e}, x\rangle = 1$, where $\mathbf{e}$ stands for a vector of one's. One refers to x as a *Pareto eigenvector* and (x, λ) as a *Pareto eigenpair*. The set of Pareto eigenvalues, denoted by $\Pi(A)$, is called the *Pareto spectrum* of A.

The constrained eigenvalue problem (1) arises in various fields of applied mathematics. For the reader's convenience we mention below two examples.

Example 1.1. A symmetric matrix A is copositive if $\langle x, Ax\rangle \geqslant 0$ for all $x \geqslant \mathbf{0}$. Checking copositivity is a difficult numerical issue addressed by many authors. As mentioned in [10], a symmetric matrix is copositive if and only if all of its Pareto eigenvalues are nonnegative.

Example 1.2. Consider a dynamical system of the form $\dot{z}(t) - Az(t) \in N_K(z(t))$, where N_K stands for the normal cone map in the sense of convex analysis. The unconstrained case $K = \mathbb{R}^n$ leads to the usual linear system $\dot{z}(t) = Az(t)$. By contrast, if the constraint set K is the nonnegative orthant of $\mathbb{R}^n$, then the above differential inclusion becomes

$$z(t) \geqslant \mathbf{0}, \quad Az(t) - \dot{z}(t) \geqslant \mathbf{0}, \quad \langle z(t), Az(t) - \dot{z}(t)\rangle = 0. \tag{2}$$

A nonzero solution to (2) is obtained by setting $z(t) = e^{\lambda t}x$, where (x, λ) is any Pareto eigenpair of A.

Further justification of the importance of the complementarity system (1) can be found in [7,20]. Numerical methods for solving (1) have been proposed in [1,11–14,17,18]. The main concern of this paper is to determine whether a given sample

$$\Xi = \{(x_1, \lambda_1), \ldots, (x_p, \lambda_p)\} \tag{3}$$

can be used to produce a matrix $A \in \mathbb{M}_n$ admitting the elements of Ξ as Pareto eigenpairs. We would like to know if A is unique and, if that is the case, how to find such a matrix. This is what we call an Inverse Pareto Eigenvalue Problem (IPEP). There are good reasons for studying this class of problems. By way of motivation we mention just one example.

Example 1.3. Let $B \in \mathbb{M}_n$ be a symmetric matrix with usual eigenvalues $\{\mu_1, \ldots, \mu_n\}$ and corresponding eigenvectors $\{u_1, \ldots, u_n\}$. The positive semidefinite symmetric matrix at minimal distance from B is given by

$$\tilde{B} = \sum_{k=1}^{n} \max\{0, \mu_k\} u_k u_k^T.$$

Consider now an arbitrary matrix $B \in \mathbb{M}_n$, not necessarily symmetric. One knows its Pareto spectrum $\Pi(B) = \{\lambda_1, \ldots, \lambda_p\}$ and a corresponding set $\{x_1, \ldots, x_p\}$ of Pareto eigenvectors. Suppose that on each λ_k one performs a certain scalar operation $\tilde{\lambda}_k = f(\lambda_k)$. Think for instance of $f(t) = \max\{0, t\}$. One produces in this way a sample

$$\{(x_1, \tilde{\lambda}_1), \ldots, (x_p, \tilde{\lambda}_p)\}$$

as in (3). One wishes to identify a matrix A associated to this sample, if there is a matrix at all. What sort of relationship there is between A and the original matrix B?

2. Geometry of the solution set of an IPEP

In the sequel we use the notation $\mathbb{N}_p = \{1, \ldots, p\}$. The Greek letter Ξ and the term *sample* are exclusively reserved for a finite set (3) whose elements satisfy the following axioms:

Table 1
Tabular representation of Ξ. The term x_k^i corresponds to the ith component of x_k.

λ_1	λ_2	$\cdots$	λ_p
x_1^1	x_2^1	$\cdots$	x_p^1
x_1^2	x_2^2	$\cdots$	x_p^2
$\cdots$	$\cdots$	x_k^i	$\cdots$
x_1^n	x_2^n	$\cdots$	x_p^n

Table 2
The set $\mathcal{A}_\Xi$ is empty, but $\mathcal{S}_\Xi$ is not.

1	4	5	16	17	20	21
1	0	1/3	0	1/5	0	1/7
0	1	2/3	0	0	1/3	2/7
0	0	0	1	4/5	2/3	4/7

$$\begin{cases} x_1, \ldots, x_p \text{ are distinct vectors in } \mathbb{R}_+^n, \\ \lambda_1, \ldots, \lambda_p \text{ are reals, not necessarily distinct,} \\ \langle \mathbf{e}, x_1 \rangle = 1, \ldots, \langle \mathbf{e}, x_p \rangle = 1. \end{cases}$$

The integer p is referred to as the size of the sample Ξ. This number is not to be confused with the *spectral size* of Ξ, which is the cardinality of the set

$$\Lambda_\Xi = \{\lambda_1, \ldots, \lambda_p\}.$$

Recall that we are allowing repetitions among the λ_k's. By contrast, the columns of the $n \times p$ matrix

$$X_\Xi = [x_1, \ldots, x_p]$$

are all different. When it comes to work with concrete examples it is convenient to represent Ξ as in Table 1.

We are interested in studying the set $\mathcal{S}_\Xi$ of all matrices A that solve the following system of linear inequalities and equalities:

$$\begin{cases} Ax_k - \lambda_k x_k \geqslant \mathbf{0} \\ \langle x_k, Ax_k - \lambda_k x_k \rangle = 0 \end{cases} \quad \text{for all } k \in \mathbb{N}_p. \tag{4}$$

In general, $\mathcal{S}_\Xi$ is a possibly empty convex polyhedron in $\mathbb{M}_n$. This polyhedron contains the affine space

$$\mathcal{A}_\Xi = \{A \in \mathbb{M}_n : Ax_k = \lambda_k x_k \text{ for all } k \in \mathbb{N}_p\},$$

which corresponds to the solution set to the classical inverse eigenvalue problem. In most of our examples the affine space $\mathcal{A}_\Xi$ is empty, but the polyhedron $\mathcal{S}_\Xi$ is not. This situation occurs typically when p is larger than n.

Example 2.1. Let Ξ be given by Table 2. Here $p = 7$ is larger than $n = 3$. Already with the first 4 elements of Ξ one realizes that $\mathcal{A}_\Xi$ is empty. By contrast, $\mathcal{S}_\Xi$ is nonempty because it contains the matrix

$$A = \begin{bmatrix} 1 & 2 & 4 \\ 2 & 4 & 8 \\ 4 & 8 & 16 \end{bmatrix}. \tag{5}$$

2.1. Inner and outer dimension of $\mathcal{S}_\Xi$

Our first result is a theorem that provides an exact estimate for

$$\dim_*(\mathcal{S}_\Xi) = \max_{\substack{\mathcal{M} \text{ affine} \\ \mathcal{M} \subset \mathcal{S}_\Xi}} \dim(\mathcal{M}), \tag{6}$$

a number called the *inner dimension* of $\mathcal{S}_\Xi$. If the set $\mathcal{S}_\Xi$ is nonempty, then the affine space achieving the maximum in (6) exists and is unique up to translation (cf. [9, Section 2.5]). We mention in passing that $\dim_*(\mathcal{S}_\Xi)$ coincides with the dimension of the largest linear subspace contained in the recession cone of $\mathcal{S}_\Xi$.

Theorem 2.2. *Let $\mathcal{S}_\Xi$ be nonempty and r be the rank of X_Ξ. Then*

$$\dim_*(\mathcal{S}_\Xi) = n(n-r).$$

In other words, $\mathcal{S}_\Xi$ contains an affine space of dimension $n(n-r)$, but not an affine space of higher dimension.

Proof. Our proof is not the shortest possible, but we take the opportunity to introduce relevant material for later discussion. We distinguish two cases:

I. *Rank deficiency*: $r < n$. Let $\{c_1, \ldots, c_{n-r}\}$ be a basis for the linear subspace of $\mathbb{R}^n$ that is orthogonal to

$$\operatorname{Im}(X_\Xi) = \operatorname{span}\{x_1, \ldots, x_p\}.$$

For each $(j, q) \in \mathbb{N}_{n-r} \times \mathbb{N}_n$ one defines $C_{j,q} \in \mathbb{M}_n$ as the matrix whose rows are all equal to $\mathbf{0}^T$, except for the qth row which is equal to c_j^T. It is not difficult to check that

$$\mathcal{C} = \{C_{j,q} : (j, q) \in \mathbb{N}_{n-r} \times \mathbb{N}_n\}$$

is a linearly independent set. Hence, $\operatorname{span}(\mathcal{C})$ has dimension $n(n-r)$. We claim the

$$\mathcal{S}_\Xi + \operatorname{span}(\mathcal{C}) = \mathcal{S}_\Xi. \tag{7}$$

One just needs to prove the nontrivial inclusion "$\subset$". Pick $A \in \mathcal{S}_\Xi$ and form any linear combination

$$C = \sum_{j=1}^{n-r} \sum_{q=1}^{n} t_{j,q} C_{j,q}.$$

Since the c_j's are in the orthogonal of $\operatorname{Im}(X_\Xi)$, one has $(A + C)x_k = Ax_k$ for all $k \in \mathbb{N}_p$. Since A belongs to $\mathcal{S}_\Xi$, so does then the matrix $A + C$. This confirms the claim (7) and proves that $\mathcal{S}_\Xi$ contains an affine space of dimension $n(n-r)$. We now prove that $\mathcal{S}_\Xi$ does not contain an affine space of higher dimension. Suppose that

$$A + \operatorname{span}\{C_1, \ldots, C_m\} \subset \mathcal{S}_\Xi$$

for some $A \in \mathbb{M}_n$ and some linearly independent set $\{C_1, \ldots, C_m\}$ in $\mathbb{M}_n$. We claim that

$$m \leqslant n(n-r). \tag{8}$$

By assumption, the affine function $t \in \mathbb{R}^m \mapsto \Phi(t) = A + \sum_{j=1}^m t_j C_j$ takes its values in $\mathcal{S}_\Xi$. In particular, A solves (4). In order to proceed further with the proof, and for latter use as well, it is helpful to reformulate (4) in a different way. We start by writing (4) component wisely:

$$\begin{cases} \langle a^i, x_k \rangle - \lambda_k x_k^i \geqslant 0 \\ x_k^i(\langle a^i, x_k \rangle - \lambda_k x_k^i) = 0 \end{cases} \quad \text{for all } (i, k) \in \mathbb{N}_n \times \mathbb{N}_p.$$

Here a^i denotes the ith column of A^T (or, equivalently, the ith row of A written as a column vector). If one introduces the index sets

$$L_0 = \{(i,k) \in \mathbb{N}_n \times \mathbb{N}_p : x_k^i = 0\},$$
$$L_1 = \{(i,k) \in \mathbb{N}_n \times \mathbb{N}_p : x_k^i > 0\},$$

then one gets the equivalent system

$$\begin{cases} \langle a^i, x_k\rangle \geqslant 0 & \text{for all } (i,k) \in L_0, \\ \langle a^i, x_k\rangle = \lambda_k x_k^i & \text{for all } (i,k) \in L_1. \end{cases} \tag{9}$$

If one endows the space $\mathbb{M}_n$ with the trace inner product $\langle C|D\rangle = \mathrm{tr}(C^T D)$, then one sees that (4) is nothing but a linear system

$$\begin{cases} \langle V_{i,k}|A\rangle \geqslant 0 & \text{for all } (i,k) \in L_0, \\ \langle V_{i,k}|A\rangle = \lambda_k x_k^i & \text{for all } (i,k) \in L_1. \end{cases} \tag{10}$$

of inequalities and equalities. Here

$$V_{i,k} = P_i^* x_k = [\mathbf{0}, \ldots, \mathbf{0}, x_k, \mathbf{0}, \ldots, \mathbf{0}]^T \tag{11}$$

is the matrix whose rows are all equal to $\mathbf{0}^T$, except for the ith row which is equal to x_k^T. The linear map $P_i^* : \mathbb{R}^n \to \mathbb{M}_n$ is the adjoint of $A \in \mathbb{M}_n \mapsto P_i(A) = a^i$. In view of the above discussion, that $\Phi(t)$ solves (4) means

$$\langle V_{i,k}|A\rangle + \sum_{j=1}^m t_j \langle V_{i,k}|C_j\rangle \geqslant 0 \quad \text{for all } (i,k) \in L_0,$$
$$\langle V_{i,k}|A\rangle + \sum_{j=1}^m t_j \langle V_{i,k}|C_j\rangle = \lambda_k x_k^i \quad \text{for all } (i,k) \in L_1.$$

Since this is true for every $t \in \mathbb{R}^m$, after simplification one gets $\langle V_{i,k}|C_j\rangle = 0$ for all $(i,k) \in L_0 \cup L_1$ and all $j \in \mathbb{N}_m$. So, each matrix in $\{C_1, \ldots, C_m\}$ is orthogonal to each matrix in

$$\underbrace{\{V_{i,k} : (i,k) \in L_0\}}_{\mathcal{V}_0} \cup \underbrace{\{V_{i,k} : (i,k) \in L_1\}}_{\mathcal{V}_1},$$

and therefore

$$m + \dim[\mathrm{span}(\mathcal{V}_0 \cup \mathcal{V}_1)] \leqslant n^2. \tag{12}$$

For completing the proof of (8) one needs to check that

$$\dim[\mathrm{span}(\mathcal{V}_0 \cup \mathcal{V}_1)] = nr,$$

but this equality follows from the special structure of the matrices $V_{i,k}$ and the fact that X_Ξ has rank equal to r.

II. *Rank completeness*: $r = n$. This case is easier because one gets

$$n(n-r) = 0,$$
$$\dim[\mathrm{span}(\mathcal{V}_0 \cup \mathcal{V}_1)] = n^2.$$

Since $\mathcal{S}_\Xi$ is nonempty, it contains an affine space of dimension 0. On the other hand, $\mathcal{S}_\Xi$ cannot contain an affine space of dimension $m \geqslant 1$ because this would contradict (12). $\square$

Table 3
$\mathcal{A}_\Xi$ is empty, but $\dim_*(\mathcal{S}_\Xi) = 3$.

2	4	7
1	0	1/2
0	1	1/2
0	0	0

Corollary 2.3. *Let $\mathcal{S}_\Xi$ be nonempty. Then $\mathcal{S}_\Xi$ is line-free if and only if* $\operatorname{rank}(X_\Xi) = n$.

Proof. As the name suggests, a nonempty convex polyhedron is line-free if it contains no line. This amounts to saying that its inner dimension is equal to 0. □

While dealing with IPEP's one usually works with samples of size larger than n. The next corollary has then a moderate interest, but we mention it for the sake of completeness.

Corollary 2.4. *For a sample Ξ, each one of the following conditions implies the next:*

(a) *The columns of X_Ξ are linearly independent.*
(b) *$\mathcal{A}_\Xi$ is nonempty.*
(c) *$\mathcal{A}_\Xi$ solves the maximization problem* (6).

Proof. (a) ⇒ (b). One must prove the existence of $A \in \mathbb{M}_n$ such that

$$A[x_1, \ldots, x_p] = [\lambda_1 x_1, \ldots, \lambda_p x_p],$$

but this is a classical inverse eigenvalue problem and the existence of such A is known.
(b) ⇒ (c). Suppose that X_Ξ has rank r. Since $\mathcal{A}_\Xi$ is nonempty, so is the set $\mathcal{S}_\Xi$. On the other hand, one has

$$\mathcal{A}_\Xi = \{A \in \mathbb{M}_n : a^i \in G^i \text{ for all } i \in \mathbb{N}_n\},$$

where each G^i is an affine space of dimension $n - r$, to wit

$$G^i = \{u \in \mathbb{R}^n : \langle x_k, u\rangle = \lambda_k x_k^i \text{ for all } k \in \mathbb{N}_p\}.$$

Hence, $\mathcal{A}_\Xi$ has dimension $n(n - r)$. It suffices now to invoke Theorem 2.2 and the fact that $\mathcal{A}_\Xi$ is an affine space contained in $\mathcal{S}_\Xi$. □

Condition (c) of Corollary 2.4 does not hold if $\mathcal{A}_\Xi$ is empty. The example below illustrates this point.

Example 2.5. Consider the sample Ξ given by Table 3. A matter of computation shows that $\mathcal{A}_\Xi$ is empty and

$$\mathcal{S}_\Xi = \left\{ \begin{bmatrix} 2 & 5 & a_{1,3} \\ 3 & 4 & a_{2,3} \\ a_{3,1} & a_{3,2} & a_{3,3} \end{bmatrix} : a_{3,1} \geqslant 0,\, a_{3,2} \geqslant 0,\, a_{1,3}, a_{2,3}, a_{3,3} \in \mathbb{R} \right\}.$$

An affine space of largest dimension contained in $\mathcal{S}_\Xi$ is

$$\begin{bmatrix} 2 & 5 & 0 \\ 3 & 4 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \left\{ \begin{bmatrix} 0 & 0 & a_{1,3} \\ 0 & 0 & a_{2,3} \\ 0 & 0 & a_{3,3} \end{bmatrix} : a_{1,3}, a_{2,3}, a_{3,3} \in \mathbb{R} \right\},$$

and therefore $\dim_*(\mathcal{S}_\Xi) = 3$.

The next proposition concerns the estimation of the *outer dimension* of $\mathcal{S}_\Xi$, which by definition is the integer

$$\dim(\mathcal{S}_\Xi) = \min_{\substack{\mathcal{M} \text{ affine} \\ \mathcal{S}_\Xi \subset \mathcal{M}}} \dim(\mathcal{M}).$$

Equivalently, $\dim(\mathcal{S}_\Xi)$ is the dimension of the affine hull of $\mathcal{S}_\Xi$. A sample Ξ is said to satisfy the *Slater condition* if there exists a matrix $\tilde{A} \in \mathbb{M}_n$ such that

$$\begin{cases} \langle V_{i,k} | \tilde{A} \rangle > 0 & \text{for all } (i,k) \in L_0 \\ \langle V_{i,k} | \tilde{A} \rangle = \lambda_k x_k^i & \text{for all } (i,k) \in L_1. \end{cases} \tag{13}$$

The use of this type of "constraint qualification assumption" is standard in convex analysis.

Proposition 2.6. *For any sample Ξ one has*

$$\dim(\mathcal{S}_\Xi) \leqslant n^2 - \dim[\operatorname{span}(\mathcal{V}_1)]. \tag{14}$$

In particular, $\mathcal{S}_\Xi$ has empty interior in $\mathbb{M}_n$. The relation (14) becomes an equality if, for instance, Ξ satisfies the Slater condition.

Proof. The second part of (10) implies that $\mathcal{S}_\Xi \subset \mathcal{H}_\Xi$, where

$$\mathcal{H}_\Xi = \{A \in \mathbb{M}_n : \langle V_{i,k} | A \rangle = \lambda_k x_k^i \text{ for all } (i,k) \in L_1\}$$

is an affine space of dimension equal to $n^2 - \dim[\operatorname{span}(\mathcal{V}_1)]$. Since the index set L_1 contains at least one element, the polyhedron $\mathcal{S}_\Xi$ has empty interior in $\mathbb{M}_n$. Suppose now that Ξ satisfies the Slater condition. If $\tilde{A} \in \mathbb{M}_n$ is as in (13), then

$$\{A \in \mathcal{H}_\Xi : \|A - \tilde{A}\| \leqslant \varepsilon\} \subset \mathcal{S}_\Xi$$

for some $\varepsilon > 0$ small enough. Hence, $\mathcal{S}_\Xi$ has nonempty interior relative to the affine space $\mathcal{H}_\Xi$. As a consequence, $\dim(\mathcal{S}_\Xi)$ is nothing but the dimension of $\mathcal{H}_\Xi$. $\square$

Example 2.7. Consider the sample Ξ given by Table 3. The Slater condition holds with

$$\tilde{A} = \begin{bmatrix} 2 & 5 & * \\ 3 & 4 & * \\ + & + & * \end{bmatrix},$$

where the entries marked with "+" are positive and marked with "$*$" are arbitrary. Hence, $\dim(\mathcal{S}_\Xi)$ is equal to the dimension of

$$\mathcal{H}_\Xi = \{A \in \mathbb{M}_n : a_{1,1} = 2,\ a_{1,2} = 5,\ a_{2,1} = 3,\ a_{2,2} = 4\},$$

that is, $\dim(\mathcal{S}_\Xi) = 3^2 - 4 = 5$.

What happens if Ξ does not satisfy the Slater condition? By using (10) and the general theory of linear inequalities one gets an explicit formula of the type

$$\dim(\mathcal{S}_\Xi) = n^2 - \dim[\operatorname{span}\{V_{i,k} : (i,k) \in L_1 \cup \widehat{L}_0\}].$$

Unfortunately, this formula involves an index set

$$\widehat{L}_0 = \{(i,k) \in L_0 : \langle V_{i,k} | A \rangle = 0 \text{ for all } A \in \mathcal{S}_\Xi\}$$

which is not always easy to identify in practice.

2.2. Bounded part and conic part of $\mathcal{S}_\Xi$

The next proposition is helpful when it comes to check whether $\mathcal{S}_\Xi$ is bounded or not.

Proposition 2.8. *If $\mathcal{S}_\Xi$ is nonempty, then the following conditions are equivalent:*

(a) *$\mathcal{S}_\Xi$ is a polytope.*
(b) *The IPEP associated to the homogeneized sample $\Xi^{\text{hom}} = \{(x_1, 0), \ldots, (x_p, 0)\}$ admits the zero matrix as unique solution.*
(c) *Any matrix W in $\mathbb{M}_n$ is expressible as linear combination $W = \sum_{i=1}^n \sum_{k=1}^p t_{i,k} V_{i,k}$ with $t_{i,k} \geqslant 0$ for $(i, k) \in L_0$ and $t_{i,k} \in \mathbb{R}$ for $(i, k) \in L_1$.*

Proof. (a) $\Leftrightarrow$ (b). Let $\text{rec}(\mathcal{S}_\Xi)$ be the recession cone of $\mathcal{S}_\Xi$. The definition of recession cone that we use is that of Rockafellar [19, Section 8]. By relying on (10) one sees that $\text{rec}(\mathcal{S}_\Xi)$ is the set of matrices $H \in \mathbb{M}_n$ such that

$$\begin{cases} \langle V_{i,k} | H \rangle \geqslant 0 \ \text{ for all } (i, k) \in L_0, \\ \langle V_{i,k} | H \rangle = 0 \ \text{ for all } (i, k) \in L_1. \end{cases}$$

The recession cone of $\mathcal{S}_\Xi$ is thus the solution set to the IPEP associated to the homogeneized sample Ξ^{hom}. In short, $\text{rec}(\mathcal{S}_\Xi) = \mathcal{S}_{\Xi^{\text{hom}}}$. The equivalence between (a) and (b) is then a consequence of the fact that a convex polyhedron is a polytope if and only if its recession cone reduces to the origin of the underlying vector space.
(b) $\Leftrightarrow$ (c). The polyhedral cone $\mathcal{V}_\Xi = \text{cone}(\mathcal{V}_0 \cup \mathcal{V}_1 \cup -\mathcal{V}_1)$ is formed with the matrices $W \in \mathbb{M}_n$ that are expressible as in (c). One can check that

$$\mathcal{S}_{\Xi^{\text{hom}}} = \{H \in \mathbb{M}_n : \langle W | H \rangle \geqslant 0 \ \text{ for all } W \in \mathcal{V}_\Xi\}$$
$$\mathcal{V}_\Xi = \{W \in \mathbb{M}_n : \langle W | H \rangle \geqslant 0 \ \text{ for all } H \in \mathcal{S}_{\Xi^{\text{hom}}}\}.$$

In other words, $\mathcal{S}_{\Xi^{\text{hom}}}$ and $\mathcal{V}_\Xi$ are mutually dual cones. This explains the equivalence between (b) and (c). □

If $\mathcal{S}_\Xi$ is nonempty and $\text{rank}(X_\Xi) = n$, then the collection $\text{ext}(\mathcal{S}_\Xi) = \{E_1, \ldots, E_q\}$ of extreme points of $\mathcal{S}_\Xi$ is nonempty and one can write

$$\mathcal{S}_\Xi = \underbrace{\text{co}[\text{ext}(\mathcal{S}_\Xi)]}_{\text{polytope}} + \underbrace{\mathcal{S}_{\Xi^{\text{hom}}}}_{\text{polyhedral cone}} \tag{15}$$

with "co" standing for the convex hull operation. The equality (15) is a particular case of a general decomposition formula for line-free convex polyhedra. We refer to the elements of $\text{ext}(\mathcal{S}_\Xi)$ as extreme solutions to the system (4).

Example 2.9. The sample Ξ given by Table 4 produces the polytope

$$\mathcal{S}_\Xi = \left\{ \begin{bmatrix} -2 & 6 & 1 \\ 5 & -1 & 1 \\ 2(1-t) & 2t & 3 \end{bmatrix} : t \in [0, 1] \right\}$$

Table 4
$\mathcal{S}_\Xi$ is a polytope with exactly two extreme points.

−2	−1	3	4	5
1	0	0	1/2	1/3
0	1	0	1/2	1/3
0	0	1	0	1/3

as solution set. The extreme solutions to the system (4) are

$$E_1 = \begin{bmatrix} -2 & 6 & 1 \\ 5 & -1 & 1 \\ 2 & 0 & 3 \end{bmatrix}, \quad E_2 = \begin{bmatrix} -2 & 6 & 1 \\ 5 & -1 & 1 \\ 0 & 2 & 3 \end{bmatrix}.$$

2.3. Computing extreme points and directions of $\mathcal{S}_\Xi$

We have not yet exploited the fact that $\mathcal{S}_\Xi$ can be identified with a Cartesian product of convex polyhedra in $\mathbb{R}^n$. Indeed, the system (9) shows that

$$A \in \mathcal{S}_\Xi \iff a^i \in C^i \quad \text{for all } i \in \mathbb{N}_n, \tag{16}$$

where each C^i is a convex polyhedron in $\mathbb{R}^n$, to wit

$$C^i = \{u \in \mathbb{R}^n : \langle x_k, u\rangle \geqslant 0\ \forall k \in L_0^i,\ \langle x_k, u\rangle = \lambda_k x_k^i\ \forall k \in L_1^i\}.$$

Here one uses the notation

$$L_0^i = \{k \in \mathbb{N}_p : x_k^i = 0\},$$
$$L_1^i = \{k \in \mathbb{N}_p : x_k^i > 0\}.$$

The equivalence (16) can be formulated in the set-theoretic form

$$\mathcal{S}_\Xi = \mathcal{I}_n^{-1}(C^1 \times \cdots \times C^n) \tag{17}$$

with $\mathcal{I}_n : \mathbb{M}_n \to (\mathbb{R}^n)^n$ being the linear isomorphism given by $\mathcal{I}_n(A) = (a^1, \ldots, a^n)$. That $\mathcal{S}_\Xi$ is isomorphic to a Cartesian product has many consequences. For instance, one can characterize the recession cone of $\mathcal{S}_\Xi$ as follows:

$$\operatorname{rec}(\mathcal{S}_\Xi) = \mathcal{I}_n^{-1}(\operatorname{rec}(C^1) \times \cdots \times \operatorname{rec}(C^n)). \tag{18}$$

In particular, $\mathcal{S}_\Xi$ is a polytope if and only if each C^i is a polytope. One can use (17) also to derive a rule for computing the extreme points of $\mathcal{S}_\Xi$.

Corollary 2.10. *Suppose that* $\operatorname{rank}(X_\Xi) = n$. *Then*

$$\operatorname{ext}(\mathcal{S}_\Xi) = \mathcal{I}_n^{-1}(\operatorname{ext}(C^1) \times \cdots \times \operatorname{ext}(C^n)). \tag{19}$$

Proof. If one of the C^i is empty, then so is $\mathcal{S}_\Xi$ and (19) holds trivially. We assume then that each C^i is nonempty, in which case also $\mathcal{S}_\Xi$ is nonempty. In view of the full rank hypothesis, the sets $\mathcal{S}_\Xi$, $C^1, \ldots, C^n$ are line-free, and therefore they possess extreme points. It suffices now to combine (17) and the general identity

$$\operatorname{ext}(C^1 \times \cdots \times C^n) = \operatorname{ext}(C^1) \times \cdots \times \operatorname{ext}(C^n)$$

for a Cartesian product of finitely many convex polyhedra. □

Example 2.11. Let Ξ be given by Table 4. Here C^1 is described by means of the linear inequalities and equalities

$$u_2 \geqslant 0, \quad u_3 \geqslant 0, \quad u_1 = -2, \quad u_1 + u_2 = 4, \quad u_1 + u_2 + u_3 = 5.$$

This yields $u_1 = -2,\ u_2 = 6,\ u_3 = 1$ as unique solution. Similarly, C^2 is given by the system

$$u_1 \geqslant 0, \quad u_3 \geqslant 0, \quad u_2 = -1, \quad u_1 + u_2 = 4, \quad u_1 + u_2 + u_3 = 5,$$

whose solution $u_1 = 5, \; u_2 = -1, \; u_3 = 1$ is also unique. Finally, C^3 is given by

$$u_1 \geqslant 0, \quad u_2 \geqslant 0, \quad u_3 = 3, \quad u_1 + u_2 + u_3 = 5.$$

Note that C^3 is a polytope with two extreme points. One gets

$$\underbrace{\left\{ \begin{bmatrix} -2 \\ 6 \\ 1 \end{bmatrix} \right\}}_{\mathrm{ext}(C^1)}, \quad \underbrace{\left\{ \begin{bmatrix} 5 \\ -1 \\ 1 \end{bmatrix} \right\}}_{\mathrm{ext}(C^2)}, \quad \underbrace{\left\{ \begin{bmatrix} 2 \\ 0 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ 3 \end{bmatrix} \right\}}_{\mathrm{ext}(C^3)}.$$

This information allows to recover both extreme solutions E_1, E_2 mentioned in Example 2.9.

Corollary 2.10 can be extended to a broader context: instead of considering only the zero-dimensional faces of $\mathcal{S}_\Xi$, one can consider the whole collection of faces of $\mathcal{S}_\Xi$. A set $\mathcal{F}$ is called a *face* of $\mathcal{S}_\Xi$ if there exists a matrix $B \in \mathbb{M}_n$ such that

$$\mathcal{F} = \mathrm{argmin}_{A \in \mathcal{S}_\Xi} \langle A|B \rangle.$$

A face of a convex polyhedron in $\mathbb{R}^n$ is defined in a similar way.

Proposition 2.12. *Let $\mathcal{S}_\Xi$ be nonempty. Then $\mathcal{F}$ is a face of $\mathcal{S}_\Xi$ if and only if there are faces $F^1, \ldots, F^n$ of $C^1, \ldots, C^n$, respectively, such that $\mathcal{F} = \mathcal{I}_n^{-1}(F^1 \times \cdots \times F^n)$.*

Proof. The proof is a matter of combining (17) and the algebra of faces for convex polyhedra (cf. [15, Lemma 2.1]). $\square$

What about the extreme directions of $\mathcal{S}_\Xi$? Let $\mathbf{O}$ denote the zero matrix in $\mathbb{M}_n$. An extreme direction of the polyhedron $\mathcal{S}_\Xi$ is a matrix $H \in \mathrm{rec}(\mathcal{S}_\Xi)\backslash\{\mathbf{O}\}$ such that $H = U + V$ with $U, V \in \mathrm{rec}(\mathcal{S}_\Xi)\backslash\{\mathbf{O}\}$, implies $H = \alpha U = \beta V$ for some $\alpha > 0$ and $\beta > 0$. In that case one refers to the half-line $\mathbb{R}_+ H$ as an extreme ray of $\mathcal{S}_\Xi$. By polyhedrality, the set

$$\{H/\|H\| \, : H \text{ is an extreme direction of } \mathcal{S}_\Xi\}$$

is finite. In other words, $\mathcal{S}_\Xi$ can have only a finite number of normalized extreme directions. The next proposition provides a method for computing the extreme directions of $\mathcal{S}_\Xi$, be them normalized or not.

Proposition 2.13. *Let $\mathcal{S}_\Xi$ be unbounded and line-free. Then the following statements are equivalent:*

(a) *$H \in \mathbb{M}_n$ is an extreme direction (respectively, normalized extreme direction) of $\mathcal{S}_\Xi$.*
(b) *There exist $i \in \mathbb{N}_n$ and an extreme direction (respectively, normalized extreme direction) $h \in \mathbb{R}^n$ of C^i such that $H = P_i^* h = [\mathbf{0}, \ldots, \mathbf{0}, h, \mathbf{0}, \ldots, \mathbf{0}]^T$.*

Proof. Consider an arbitrary collection $\{K_1, \ldots, K_m\}$ of pointed polyhedral cones in $\mathbb{R}^n$. The m-tuple $(h_1, \ldots, h_m)$ is an extreme direction of $K_1 \times \cdots \times K_m$ if and only if there exist $i_0 \in \mathbb{N}_m$ and an extreme direction h of K_{i_0} such that

$$h_i = \begin{cases} h & \text{if } i = i_0, \\ \mathbf{0} & \text{if } i \neq i_0. \end{cases}$$

This is a general principle for computing extreme directions in a Cartesian product of pointed polyhedral cones. Now, the fact that $\mathcal{S}_\Xi$ is line-free implies that each C^i is line-free. Hence, each set $\mathrm{rec}(C^i)$ is a

Table 5
$\mathcal{S}_\Xi$ is unbounded and line-free. It has one extreme point and three extreme rays.

0	2	4	5
1	0	1/2	1/2
0	1	0	1/2
0	0	1/2	0

pointed polyhedral cone. For proving the proposition it suffices then to combine the above mentioned principle and the formula (18). $\square$

Example 2.14. Consider the sample Ξ given by Table 5.
One gets

$$C^1 = \{(0,5,4)\},$$
$$C^2 = \{(3,2,\gamma) : \gamma \geqslant -3\},$$
$$C^3 = \{(\alpha,\beta,4-\alpha) : \alpha \geqslant 0, \beta \geqslant 0\}.$$

Note that C^1 has no extreme rays, C^2 has one extreme ray, namely $\mathbb{R}_+(0,0,1)$, and C^3 has two extreme rays, namely, $\mathbb{R}_+(0,1,0)$ and $\mathbb{R}_+(1,0,-1)$. Hence, the solution set

$$\mathcal{S}_\Xi = \left\{ \begin{bmatrix} 0 & 5 & 4 \\ 3 & 2 & \gamma \\ \alpha & \beta & 4-\alpha \end{bmatrix} : \gamma \geqslant -3, \alpha \geqslant 0, \beta \geqslant 0 \right\}$$

has three extreme rays, and these are given by the matrices

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & -1 \end{bmatrix}.$$

There is a reach literature devoted to the problem of computing extreme points and faces of convex polyhedra in $\mathbb{R}^n$. Numerical methods are suggested in [3,4,16] and in other places. The extreme directions of a convex polyhedron in $\mathbb{R}^n$ can be computed with the help of Weyl's method (cf. [2, Theorem 3.34]) or with any other technique (cf. [5,16,21]).

3. Consistency, discriminability, and exhaustivity

A sample Ξ is called *consistent* (respectively, *discriminating*) if the linear system (4) admits at least (respectively, at most) one solution. If Ξ is consistent and discriminating, then the unique solution to (4) is denoted by A_Ξ. Below we address a list of easy facts concerning consistency and discriminability.

- Let $n \geqslant 2$. A sample with one or two elements is always consistent, with three or more elements could be inconsistent.
- The spectral size of a consistent sample Ξ cannot exceed the integer [2]

$$\pi_n = \max_{A \in \mathbb{M}_n} \operatorname{card}[\Pi(A)].$$

[2] Beware that π_n grows very rapidly with n. We have been able to prove that $\pi_n \geqslant 3(2^{n-1}-1)$. The proof of this inequality is quite long and will be presented in a forthcoming technical note. Note that, for instance, a matrix of order 20 could have more than 1.5 million Pareto eigenvalues!

Table 6
A sample Ξ that is inconsistent.

−1	1	3	5
1/2	1/3	1/5	1/7
1/2	2/3	0	2/7
0	0	4/5	4/7

- Let $a > 0$ and $b \in \mathbb{R}$. If a sample Ξ is consistent (respectively, discriminating), then the sample $\{(x_1, a\lambda_1 + b), \ldots, (x_p, a\lambda_p + b)\}$ is also consistent (respectively, discriminating).
- Let P be a permutation matrix of order n. If a sample Ξ is consistent (respectively, discriminating), then the sample $\{(Px_1, \lambda_1), \ldots, (Px_p, \lambda_p)\}$ is also consistent (respectively, discriminating).

A necessary and sufficient condition for consistency is established in Theorem 3.1. The result is written in a negative form, i.e., one characterizes inconsistency. The notation K^i_Ξ stands for the polyhedral cone in $\mathbb{R}^{n+1}$ generated by the vectors

$$\left\{ \begin{bmatrix} x_k \\ 0 \end{bmatrix} : k \in L^i_0 \right\} \bigcup \left\{ \pm \begin{bmatrix} x_k \\ \lambda_k x^i_k \end{bmatrix} : k \in L^i_1 \right\}.$$

Theorem 3.1. *A sample Ξ is inconsistent if and only if*

$$\begin{bmatrix} \mathbf{0} \\ 1 \end{bmatrix} \in K^i_\Xi \quad \textit{for some } i \in \mathbb{N}_n. \tag{20}$$

Proof. A sample Ξ is consistent if and only if the system (4) is feasible in $\mathbb{M}_n$. Equivalently, for each $i \in \mathbb{N}_n$ the system

$$\begin{cases} \langle x_k, u \rangle \geqslant 0 & \text{for all } k \in L^i_0, \\ \langle x_k, u \rangle = \lambda_k x^i_k & \text{for all } k \in L^i_1 \end{cases} \tag{21}$$

is feasible in $\mathbb{R}^n$. But, by applying the Gale alternative theorem (cf. [8, Chapter 3]), one sees that the feasibility of (21) is equivalent to the unfeasibility of

$$\begin{bmatrix} \mathbf{0} \\ 1 \end{bmatrix} = \sum_{k \in L^i_0} \alpha_k \begin{bmatrix} x_k \\ 0 \end{bmatrix} + \sum_{k \in L^i_1} \beta_k \begin{bmatrix} x_k \\ \lambda_k x^i_k \end{bmatrix}, \quad \alpha_k \geqslant 0 \text{ for } k \in L^i_0, \quad \beta_k \in \mathbb{R} \text{ for } k \in L^i_1.$$

This completes the proof. □

Example 3.2. Let Ξ be given by Table 6. There are no zero entries in the first row of X_Ξ. Hence, K^1_Ξ is a linear space. More precisely,

$$K^1_\Xi = \operatorname{span} \left\{ \begin{bmatrix} 1/2 \\ 1/2 \\ 0 \\ -1/2 \end{bmatrix}, \begin{bmatrix} 1/3 \\ 2/3 \\ 0 \\ 1/3 \end{bmatrix}, \begin{bmatrix} 1/5 \\ 0 \\ 4/5 \\ 3/5 \end{bmatrix}, \begin{bmatrix} 1/7 \\ 2/7 \\ 4/7 \\ 5/7 \end{bmatrix} \right\} = \mathbb{R}^4.$$

Since (20) holds for $i = 1$, the sample Ξ is inconsistent. There is no need to examine the remaining rows of X_Ξ.

The next two propositions address the issue of discriminability.

Proposition 3.3. *Let Ξ be a consistent sample. For Ξ to be discriminating it is necessary that* $\mathrm{rank}(X_\Xi) = n$.

Proof. Suppose that $r = \mathrm{rank}(X_\Xi)$ is smaller than n. Then the solution set $\mathcal{S}_\Xi$ contains a line. In fact, it contains an affine space of dimension $n(n-r) \geqslant 1$. Hence, Ξ fails to be discriminating. □

Proposition 3.4. *The following conditions are equivalent and imply that the sample Ξ is discriminating:*

(a) $\dim[\mathrm{span}(\mathcal{V}_1)] = n^2$.
(b) $\dim[\mathrm{span}\{x_k : k \in L_1^i\}] = n$ *for all* $i \in \mathbb{N}_n$.

Proof. The equivalence between (a) and (b) follows from the structure (11) of the matrices $V_{i,k}$. An easily computable upper bound for $\dim(\mathcal{S}_\Xi)$ has been proposed in Proposition 2.6. Such upper bound shows that (a) implies the discriminability of Ξ. □

If Ξ is consistent, then by using (17) one can show that $\dim(\mathcal{S}_\Xi) = \sum_{i=1}^n \dim(C^i)$. Computing the outer dimension of each C^i can be difficult and expensive, even if C^i is a convex polyhedron in $\mathbb{R}^n$. However, one can easily check that

$$\dim(C^i) \leqslant n - \dim[\mathrm{span}\{x_k : k \in L_1^i\}].$$

The next proposition is highly specialized and therefore its interest is rather limited. However, such proposition could be the starting point toward a more general formulation. The symbol "ri" stands for relative interior and $\{e_1, \ldots, e_n\}$ is the canonical basis of $\mathbb{R}^n$.

Proposition 3.5. *Let Ξ be a sample satisfying the hypothesis*

$$\forall\,(i,j) \in \mathbb{N}_n \times \mathbb{N}_n,\ \exists\, k \in \mathbb{N}_p \text{ such that } x_k \in \mathrm{ri}(\mathrm{co}\{e_i, e_j\}). \tag{22}$$

Then Ξ is discriminating. Furthermore, the unique element of $\mathcal{S}_\Xi$ (if any) is a matrix whose off-diagonal entries are nonnegative.

Proof. Suppose that Ξ is consistent, otherwise there is nothing to prove. Let $A \in \mathcal{S}_\Xi$. By taking $j = i$ in (22), one sees that the canonical vectors of $\mathbb{R}^n$ are all to be found among the columns of X_Ξ, i.e.,

$$\forall\, i \in \mathbb{N}_n,\ \exists\, k \in \mathbb{N}_p \text{ such that } x_k = e_i. \tag{23}$$

Let k_i be the unique $k \in \mathbb{N}_p$ such that $x_k = e_i$. Then

$$a_{i,i} = \lambda_{k_i} \quad \text{for all } i \in \mathbb{N}_n. \tag{24}$$

Hence, the diagonal entries of A are fully determined by the sample. We now fix a pair $(i,j) \in \mathbb{N}_n \times \mathbb{N}_n$, with $i \neq j$, and compute the the off-diagonal entries $a_{i,j}$ and $a_{j,i}$. By (23) one knows already that $a_{i,j} \geqslant 0$ and $a_{j,i} \geqslant 0$. Pick any k as in (22). Note that k may not be unique. Since A solves (4), one has

$$a_{i,i}x_k^i + a_{i,j}x_k^j = \lambda_k x_k^i,$$
$$a_{j,i}x_k^i + a_{j,j}x_k^j = \lambda_k x_k^j.$$

Plugging (24) into the above system and simplifying, one obtains

$$a_{i,j} = (\lambda_k - \lambda_{k_i})(x_k^i / x_k^j),$$
$$a_{j,i} = (\lambda_k - \lambda_{k_j})(x_k^j / x_k^i).$$

Thus, the off-diagonal entries are also determined by the sample. □

Table 7
Ξ is weakly exhaustive, but not exhaustive.

−1	0	2	4	5
0	1	0	3/7	1/2
3/7	0	1	0	1/2
4/7	0	0	4/7	0

As a first generalization of Proposition 3.5 one obtains a result concerning the set

$$\mathcal{S}_\Xi^J = \{A^J : A \in \mathcal{S}_\Xi\},$$

where $A^J = [a_{i,j}]_{i,j\in J}$ denotes the principal submatrix of A with entries indexed by $J \subset \mathbb{N}_n$. Let $|J|$ denote the cardinality of J. The next proposition is of interest if one wishes to identify only a certain portion of a solution $A \in \mathcal{S}_\Xi$. The proof is as in Proposition 3.5, and therefore it is omitted.

Proposition 3.6. *Let Ξ be a sample satisfying*

$$\forall\, (i,j) \in J \times J, \exists\, k \in \mathbb{N}_p \text{ such that } x_k \in \mathrm{ri}(\mathrm{co}\{e_i, e_j\}) \tag{25}$$

for a given $J \subset \mathbb{N}_n$. Then $\mathcal{S}_\Xi^J$ contains at most one element. Furthermore, the unique element of $\mathcal{S}_\Xi^J$ (if any) is a matrix of order $|J|$ whose off-diagonal entries are nonnegative.

Example 3.7. The sample Ξ given by Table 4 does not satisfy the hypothesis (22). However, the condition (25) holds with $J = \{1, 2\}$. This explains why

$$\mathcal{S}_\Xi^J = \left\{ \begin{bmatrix} -2 & 6 & 1 \\ 5 & -1 & 1 \\ 2(1-t) & 2t & 3 \end{bmatrix}^J : t \in [0,1] \right\} = \left\{ \begin{bmatrix} -2 & 6 \\ 5 & -1 \end{bmatrix} \right\}$$

has one element at most.

Consider a solution A to the IPEP associated to a consistent sample Ξ. It seems strange at first sight, but the matrix A may have some Pareto eigenvalues that are not in Λ_Ξ. In other words, the set Λ_Ξ could be strictly included in $\Pi(A)$. This observation motivates the next definition.

Definition 3.8. A consistent sample Ξ is called exhaustive if $\Pi(A) = \Lambda_\Xi$ for all $A \in \mathcal{S}_\Xi$ and weakly exhaustive if $\Pi(A) = \Lambda_\Xi$ for some $A \in \mathcal{S}_\Xi$.

Of course, exhaustivity implies weak exhaustivity. The next example shows that the reverse implication is not always true.

Example 3.9. Let Ξ given by Table 7. A solution to the corresponding IPEP is a matrix of the form

$$A(t) = \begin{bmatrix} 0 & 5 & 3 \\ 3 & 2 & -9/4 \\ t/3 & (t-20)/3 & (16-t)/4 \end{bmatrix}$$

with $t \geqslant 20$. A matter of computation shows that

$$\Pi(A(20)) = \{-1, 0, 2, 4, 5\} = \Lambda_\Xi,$$

$$\Pi(A(44)) = \{-4, -1, 0, 2, 4, 5\} \neq \Lambda_\Xi.$$

Hence, Ξ is weakly exhaustive, but not exhaustive.

Proposition 3.10. *Let Ξ be consistent. Then there exists a sample Ξ' that is exhaustive and such that $\Xi \subset \Xi'$.*

Proof. If $\mathrm{card}(\Lambda_\Xi) = \pi_n$, then the sample Ξ is necessarily exhaustive. Suppose then that $\mathrm{card}(\Lambda_\Xi) < \pi_n$. If Ξ is already exhaustive, then we are done. Otherwise there is a matrix $A_1 \in \mathcal{S}_\Xi$ such that $\Pi(A_1)$ is larger than Λ_Ξ. We pick a scalar $\mu_1 \in \Pi(A_1)\backslash\Lambda_\Xi$ and compute a Pareto eigenvector y_1 of A_1 associated to μ_1. If the expanded sample

$$\Xi_1 = \Xi \cup \{(y_1, \mu_1)\}$$

is exhaustive, then we stop. Otherwise, we apply the same procedure to Ξ_1 in order to get a new sample

$$\Xi_2 = \Xi_1 \cup \{(y_2, \mu_2)\},$$

and so on. In this way one generates a collection of samples $\Xi \subset \Xi_1 \subset \Xi_2 \subset \cdots$ such that

$$\mathrm{card}(\Lambda_{\Xi_q}) = \mathrm{card}(\Lambda_\Xi) + q$$

for all $q \geqslant 1$. The above process cannot continue forever: after a finite number of iterations one gets $\mathrm{card}(\Lambda_{\Xi_q}) = \pi_n$, and in such a case the sample Ξ_q is exhaustive. $\square$

Exhaustivity is a notion that is somehow antagonic to irredundancy. A consistent sample Ξ is called *redundant* if there exists a sample Ξ' strictly contained in Ξ and such that $\mathcal{S}_{\Xi'} = \mathcal{S}_\Xi$. Exhaustivity forces the spectral size of Ξ to be rather large, whereas irredundancy forces the cardinality of Ξ to be rather small.

Example 3.11. The sample Ξ given by Table 2 is consistent and discriminating. The matrix A_Ξ is given by (5). Since $\Pi(A_\Xi) = \{1, 4, 5, 16, 17, 20, 21\} = \Lambda_\Xi$, the sample Ξ is exhaustive. Note that Ξ is redundant because if one drops the last column of Table 2, then one gets a subsample Ξ' that still produces (5) as unique solution. Of course, Ξ' is not exhaustive because $\Lambda_{\Xi'}$ does not capture the Pareto eigenvalue 21.

3.1. By way of conclusion

An ideal situation occurs when the solution set $\mathcal{S}_\Xi$ is a singleton. What to do in case of inconsistency or non-discriminability? In order to deal with inconsistency we suggest to represent

$$\mathcal{S}_\Xi = \{A \in \mathbb{M}_n : F(A) = \mathbf{0}\}$$

as the root set of a certain vector-valued function F, and then compute a matrix $\bar{A}$ that minimizes the associated residual function

$$A \in \mathbb{M}_n \mapsto \mathrm{res}(A) = \|F(A)\|^2.$$

One refers to $\bar{A}$ as a least-residual solution to the IPEP. Of course, the concept of least-residual solution depends on the function F. Somebody familiar with the theory of complementary problems knows that a natural representation of $\mathcal{S}_\Xi$ as root set is

$$\mathcal{S}_\Xi = \{A \in \mathbb{M}_n : \Phi(x_k, Ax_k - \lambda_k x_k) = \mathbf{0} \text{ for all } k \in \mathbb{N}_p\},$$

where $\Phi : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ denotes the Fischer–Burmeister complementarity function for the cone $\mathbb{R}^n_+$. The minimization of the residue

$$\mathrm{res}(A) = \sum_{k=1}^{p} \|\Phi(x_k, Ax_k - \lambda_k x_k)\|^2$$

can be achieved by using techniques of nonsmooth optimization (cf. [6]).

In order to deal with non-discriminability there are several options. The choice of a particular matrix $\bar{A}$ from the solution set $\mathcal{S}_\Xi$ is essentially dictated by the context:

(i) If a solution to the IPEP is sought in a prescribed polyhedral subset $\mathcal{P}$ of $\mathbb{M}_n$, then the linear constraints defining $\mathcal{P}$ can be incorporated to the system (4). By proceeding in this way one preserves polyhedrality and reduces the size of $\mathcal{S}_\Xi$. Think for instance of the case of an IPEP that must be solved by a nonnegative matrix. At a more sophisticated level, one may think of an IPEP that must be solved by a symmetric tridiagonal matrix. Of course, nonpolyhedral constraints (like copositivity or positive semidefiniteness) fall beyond the context of our work.

(ii) Suppose that $\mathcal{S}_\Xi$ is a polytope (cf. Proposition 2.8). It is helpful in such a case to identify the extreme points $\{E_1, \ldots, E_q\}$ of $\mathcal{S}_\Xi$. Any convex combination of these extreme points is a solution of the IPEP. For instance, one may take $\bar{A}$ as the baricenter of the polytope. If one prefers instead a matrix that is small in norm, then one takes $\bar{A}$ as the least-norm element of $\mathcal{S}_\Xi$.

(iii) Suppose that we are searching for a solution to the IPEP that is near a target matrix $B \in \mathbb{M}_n$. Think for instance of Example 1.3. In such a case it is reasonable to choose $\bar{A}$ as the element of $\mathcal{S}_\Xi$ at minimal distance from B.

Concerning the item (i), a word of caution is in order. As seen in (17), up to linear isomorphism, $\mathcal{S}_\Xi$ is a Cartesian product of polyhedral sets in $\mathbb{R}^n$. This property, called Cartesian Product Representability (CPR) is very useful when it comes to practical computations. The restricted solution set $\mathcal{P} \cap \mathcal{S}_\Xi$ it still polyhedral, but may not enjoy the CPR property. For instance, $\mathcal{P} \cap \mathcal{S}_\Xi$ enjoys the CPR property if $\mathcal{P}$ is the set of nonnegative matrices, but not if $\mathcal{P}$ is the set of symmetric tridiagonal matrices.

References

[1] S. Adly, A. Seeger, A nonsmooth algorithm for cone-constrained eigenvalue problems, Comput. Optim. Appl. 49 (2) (2011) 299–318.
[2] C. Aliprantis, R. Tourky, Cones and Duality, American Math. Society, Providence, RI, 2007.
[3] C.A. Burdet, Generating all the faces of a polyhedron, SIAM J. Appl. Math. 26 (1974) 479–489.
[4] P.C. Chen, P. Hansen, B. Jaumard, On-line and off-line vertex enumeration by adjacency lists, Oper. Res. Lett. 10 (1991) 403–409.
[5] E.R. Davidson, W.B. McRae, An algorithm for the extreme rays of a pointed convex polyhedral cone, SIAM J. Comput. 2 (1973) 281–293.
[6] F. Facchinei, J.P. Pang, Finite-Dimensional Variational Inequalities and Complementarity Problems, vol. II, Springer-Verlag, New York, 2003.
[7] I.N. Figueiredo, J.J. Júdice, J.A.C. Martins, A. Pinto da Costa, A complementarity eigenproblem in the stability of finite dimensional elastic systems with frictional contact, in: M. Ferris, O. Mangasarian, J.S. Pang (Eds.), Complementarity: Applications, Algorithms and Extensions, Kluwer Acad. Publ., Dordrecht, 1999, pp. 67–83.
[8] M.A. Goberna, M.A. López, Linear Semi-infinite Optimization, John Wiley, Chichester, 1998.
[9] B. Grünbaum, Convex Polytopes, second ed., Springer-Verlag, New York, 2003.
[10] J.B. Hiriart-Urruty, A. Seeger, A variational approach to copositive matrices, SIAM Rev. 52 (2010) 593–629.
[11] C. Humes, J.J. Júdice, M. Queiroz, The symmetric eigenvalue complementarity problem, Math. Comput. 73 (2004) 1849–1863.
[12] J.J. Júdice, M. Raydan, S.S. Rosa, S.A. Santos, On the solution of the symmetric eigenvalue complementarity problem by the spectral projected gradient algorithm, Numer. Algorithms 47 (2008) 391–407.
[13] J.J. Júdice, I.M. Ribeiro, S.S. Rosa, H.D. Sherali, On the asymmetric eigenvalue complementarity problem, Optim. Methods Softw. 24 (2009) 549–568.
[14] J.J. Júdice, I.M. Ribeiro, H.D. Sherali, The eigenvalue complementarity problem, Comput. Optim. Appl. 37 (2007) 139–156.
[15] N.T.B. Kim, D.T. Luc, Normal cones to a polyhedral convex set and generating efficient faces in linear multiobjective programming, Acta Math. Vietnam. 25 (2000) 101–124.
[16] A. Omar, Finding all extreme points and extreme rays of a convex polyhedral set, Ekonom.-Mat. Obzor 13 (1977) 331–342.
[17] A. Pinto da Costa, A. Seeger, Numerical resolution of cone-constrained eigenvalue problems, Comput. Appl. Math. 28 (2009) 37–61.
[18] A. Pinto da Costa, A. Seeger, Cone-constrained eigenvalue problems: theory and algorithms, Comput. Optim. Appl. 45 (2010) 25–57.
[19] R.T. Rockafellar, Convex Analysis, Princeton Univ. Press, Princeton, 1970.
[20] A. Seeger, Eigenvalue analysis of equilibrium processes defined by linear complementarity conditions, Linear Algebra Appl. 292 (1999) 1–14.
[21] Z.L. Wei, B.K. Zhang, An algorithm for computing all extreme directions for the general form of a cone, J. Numer. Methods Comput. Appl. 6 (1985) 223–234.